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Aspects of Differential Geometry and Tensor Calculus in Anholonomic Configuration Space

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14. ABSTRACT

In the context of finite deformation mechanics, a tangent mapping is anholonomic over some domain when it is not a gradient of a motion; conversely, a deformation gradient is holonomic when it is integrable to a motion field everywhere in that domain. This brief report addresses covariant differentiation for four possible choices of basis vectors in anholonomic space. As an example from continuum physics, the theory is applied towards description of divergence of the heat flux.

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Aspects of differential geometry and tensor calculus in anholonomic configuration space

J.D. Clayton

In the context of finite deformation mechanics, a tangent mapping is anholonomic over some domain when it is not a gradient of a motion; conversely, a deformation gradient is holonomic when it is integrable to a motion field everywhere in that domain. This brief report addresses covariant differentiation for four possible choices of basis vectors in anholonomic space. As an example from continuum physics, the theory is applied towards description of divergence of the heat flux. An extensive treatment of anholonomic mathematics can be found in a recent article [1]; however, this report includes material not found in [1], and vice-versa.

As suggested by Schouten [2], consideration of differential geometry of anholonomic spaces dates back to at least 1926 [3]. Many important identities are derived in [2, 4]. Various coordinate systems and associated metric tensors in anholonomic space are considered in [5], with particular focus on convected basis and Cartesian representations. Correspondences among mathematical objects from differential geometry and their continuum physical counterparts in defect field theory of crystals are described at length in a more recent monograph [6].

The present description is limited to the time-independent case, such that spatial coordinates x^a are related to reference coordinates X^A by one-to-one and at least twice-differentiable mappings $x^a(X)$ and $X^A(x)$, with X a material particle and x its spatial representation. Let the usual holonomic deformation gradient $\boldsymbol{F}(X)$ be decomposed multiplicatively as

$$(1) \qquad \boldsymbol{F} = \bar{\boldsymbol{F}}\tilde{\boldsymbol{F}}, \qquad F^a_{.A} = \bar{F}^a_{.\alpha}\tilde{F}^\alpha_{.A};$$

$$(2) \qquad \boldsymbol{F} = \partial_A x^a \, \boldsymbol{g}_a \otimes \boldsymbol{G}^A, \qquad \bar{\boldsymbol{F}} = \bar{F}^a_{.\alpha} \boldsymbol{g}_a \otimes \tilde{\boldsymbol{g}}^\alpha, \qquad \tilde{\boldsymbol{F}} = \tilde{F}^\alpha_{.A}\tilde{\boldsymbol{g}}_\alpha \otimes \boldsymbol{G}^A.$$

Denoting $\partial_A = \partial/\partial X^A$ and $\partial_a = \partial/\partial x^a$, partial differentiation proceeds as

$$(3) \qquad \partial_\alpha(\cdot) \overset{\text{def}}{=} \bar{F}^a_{.\alpha}\partial_a(\cdot) = \tilde{F}^{-1A}_{.\alpha}\partial_A(\cdot), \qquad \partial_A(\cdot) = \partial_A x^a \partial_a(\cdot) = F^a_{.A}\partial_a(\cdot).$$

Attention is restricted to a simply connected domain in reference and current configurations such that $\{X^A\}$ and $\{x^a\}$ are global coordinate charts. Let indices in brackets be skew, e.g., $A_{[AB]} \overset{\text{def}}{=} \frac{1}{2}(A_{AB} - A_{BA})$. Since X^A and x^a are holonomic coordinates,

$$(4) \qquad \partial_{[A}\partial_{B]}(\cdot) = 0, \quad \partial_{[a}\partial_{b]}(\cdot) = 0; \quad \partial_{[A}F^a_{.B]} = 0, \quad \partial_{[a}F^{-1A}_{.b]} = 0.$$

Similar identities do not always hold for $\partial_\alpha(\cdot)$ since $\bar{\boldsymbol{F}}^{-1}$ and $\tilde{\boldsymbol{F}}$ are not necessarily integrable functions of x^a or X^A. Anholonomic object $\boldsymbol{\kappa}$ obeys [1, 2]

$$(5) \qquad \kappa^{\cdot\cdot\alpha}_{\beta\chi} \overset{\text{def}}{=} -\bar{F}^{-1\alpha}_{.a}\partial_{[\beta}\bar{F}^a_{.\chi]} = -\tilde{F}^\alpha_{.A}\partial_{[\beta}\tilde{F}^{-1A}_{.\chi]} = -\kappa^{\cdot\cdot\alpha}_{\chi\beta};$$

$$(6) \qquad \partial_{[\alpha}\partial_{\beta]}(\cdot) = -\kappa^{\cdot\cdot\chi}_{\alpha\beta}\partial_\chi(\cdot).$$

Holonomic charts $\{\tilde{x}^\alpha(X)\}$ [or $\{\tilde{x}^\alpha(x)\}$] exist in a one-to-one fashion with X or x if and only if $\kappa^{\cdot\cdot\alpha}_{\beta\chi} = 0$ throughout a simply connected domain.

Let $\boldsymbol{A}$ be a generic differentiable tensor field. Covariant differentiation in anholonomic space is defined as

(7) $\quad \nabla_\nu A^{\alpha...\phi}_{\gamma...\mu} \stackrel{\text{def}}{=} \partial_\nu A^{\alpha...\phi}_{\gamma...\mu} + \Gamma^{..\alpha}_{\nu\rho} A^{\rho...\phi}_{\gamma...\mu} + \cdots + \Gamma^{..\phi}_{\nu\rho} A^{\alpha...\rho}_{\gamma...\mu} - \Gamma^{..\rho}_{\nu\gamma} A^{\alpha...\phi}_{\rho...\mu} - \cdots - \Gamma^{..\rho}_{\nu\mu} A^{\alpha...\phi}_{\gamma...\rho}.$

Connection coefficients can be expressed in general form as [2]

(8) $\qquad \Gamma^{..\alpha}_{\beta\chi} = \frac{1}{2}\tilde{g}^{\alpha\delta}(\partial_{\{\beta}\tilde{g}_{\delta\chi\}} - 2T_{\{\beta\delta\chi\}} + 2\kappa_{\{\beta\delta\chi\}} + M_{\{\beta\delta\chi\}}),$

where $\tilde{g}^{\alpha\chi}\tilde{g}_{\chi\beta} = \delta^\alpha_\beta$ and the following definitions apply:

(9) $\qquad (\cdot)_{\{\alpha\beta\chi\}} \stackrel{\text{def}}{=} (\cdot)_{\alpha\beta\chi} - (\cdot)_{\beta\chi\alpha} + (\cdot)_{\chi\alpha\beta}, \qquad (\cdot)_{\beta\chi\delta} \stackrel{\text{def}}{=} (\cdot)^{..\alpha}_{\beta\chi}\tilde{g}_{\delta\alpha};$

(10) $\qquad \tilde{g}_{\alpha\beta} \stackrel{\text{def}}{=} \tilde{\boldsymbol{g}}_\alpha \cdot \tilde{\boldsymbol{g}}_\beta, \quad T^{..\alpha}_{\beta\chi} \stackrel{\text{def}}{=} \Gamma^{..\alpha}_{[\beta\chi]} + \kappa^{..\alpha}_{\beta\chi}, \quad M_{\alpha\beta\chi} \stackrel{\text{def}}{=} -\nabla_\alpha \tilde{g}_{\beta\chi}.$

In this report attention is restricted to metric connections so that $M_{\alpha\beta\chi} = 0$ and

(11) $\qquad \partial_\alpha \tilde{\boldsymbol{g}}_\beta = \Gamma^{..\chi}_{\alpha\beta}\tilde{\boldsymbol{g}}_\chi, \qquad \partial_\alpha\tilde{\boldsymbol{g}}^\beta = -\Gamma^{..\beta}_{\alpha\chi}\tilde{\boldsymbol{g}}^\chi;$

(12) $\qquad \partial_\alpha \ln\sqrt{\tilde{g}} = \Gamma^{..\beta}_{\alpha\beta}, \qquad \nabla_\alpha\epsilon_{\beta\chi\delta} = \partial_\alpha\epsilon_{\beta\chi\delta} - \Gamma^{..\phi}_{\alpha\phi}\epsilon_{\beta\chi\delta} = 0;$

(13) $\qquad \tilde{g} \stackrel{\text{def}}{=} \det(\tilde{g}_{\alpha\beta}), \qquad \epsilon_{\alpha\beta\chi} \stackrel{\text{def}}{=} \sqrt{\tilde{g}}e_{\alpha\beta\chi}, \qquad \epsilon^{\alpha\beta\chi} \stackrel{\text{def}}{=} (1/\sqrt{\tilde{g}})e^{\alpha\beta\chi}.$

The covariant derivative of a generic differentiable vector field $\boldsymbol{V} = V^\alpha\tilde{\boldsymbol{g}}_\alpha$ is then

(14) $\qquad \nabla\boldsymbol{V} = \partial_\beta\boldsymbol{V} \otimes \tilde{\boldsymbol{g}}^\beta = (\partial_\beta V^\alpha + \Gamma^{..\alpha}_{\beta\chi}V^\chi)\tilde{\boldsymbol{g}}_\alpha \otimes \tilde{\boldsymbol{g}}^\beta.$

Total covariant derivatives of two-point tangent mappings $\tilde{\boldsymbol{F}}$ and $\bar{\boldsymbol{F}}^{-1}$ are [1, 6]

(15) $\qquad \tilde{F}^\alpha_{.A:B} \stackrel{\text{def}}{=} \partial_B \tilde{F}^\alpha_{.A} + \Gamma^{..\alpha}_{\beta\chi}\tilde{F}^\beta_{.B}\tilde{F}^\chi_{.A} - \Gamma^{..C}_{BA}\tilde{F}^\alpha_{.C} = \tilde{F}^\alpha_{.A:\beta}\tilde{F}^\beta_{.B};$

(16) $\qquad \bar{F}^{-1\alpha}_{.a:b} \stackrel{\text{def}}{=} \partial_b \bar{F}^{-1\alpha}_{.a} + \Gamma^{..\alpha}_{\beta\chi}\bar{F}^{-1\beta}_{.b}\bar{F}^{-1\chi}_{.a} - \Gamma^{..c}_{ba}\bar{F}^{-1\alpha}_{.c} = \bar{F}^{-1\alpha}_{.a:\beta}\bar{F}^{-1\beta}_{.b}.$

Metrics and Levi-Civita connections in reference and current configurations are

(17) $\qquad G_{AB} \stackrel{\text{def}}{=} \boldsymbol{G}_A \cdot \boldsymbol{G}_B = \partial_A\boldsymbol{X} \cdot \partial_B\boldsymbol{X}, \qquad \Gamma^{..A}_{BC} \stackrel{\text{def}}{=} \frac{1}{2}G^{AD}\partial_{\{B}G_{DC\}};$

(18) $\qquad g_{ab} \stackrel{\text{def}}{=} \boldsymbol{g}_a \cdot \boldsymbol{g}_b = \partial_a\boldsymbol{x} \cdot \partial_b\boldsymbol{x}, \qquad \Gamma^{..a}_{bc} \stackrel{\text{def}}{=} \frac{1}{2}g^{ad}\partial_{\{b}g_{dc\}}.$

Letting $g = \det(g_{ab})$ and $G = \det(G_{AB})$, Jacobian determinants are [5, 6]

(19) $\quad J = \sqrt{g/G}\det(\partial_A x^a) = \bar{J}\tilde{J}, \quad \bar{J} = \sqrt{g/\tilde{g}}\det(\bar{F}^a_{.\alpha}), \quad \tilde{J} = \sqrt{\tilde{g}/G}\det(\tilde{F}^\alpha_{.A}).$

Piola's identities for possibly anholonomic Jacobian determinants are then [1, 4, 6]

(20) $\quad (\tilde{J}\tilde{F}^{-1A}_{.\alpha})_{:A} = \epsilon_{\alpha\beta\chi}\epsilon^{ABC}\tilde{F}^\beta_{.A}\tilde{F}^\chi_{.[B:C]}, \quad (\bar{J}^{-1}\bar{F}^a_{.\alpha})_{:a} = \epsilon_{\alpha\beta\chi}\epsilon^{abc}\bar{F}^{-1\beta}_{.a}\bar{F}^{-1\chi}_{.[b:c]}.$

Let $\bar{\boldsymbol{q}} = \bar{q}^\alpha\tilde{\boldsymbol{g}}_\alpha$ denote the heat flux vector referred to anholonomic space, let $k^{\alpha\beta}$ denote a covariant constant positive semi-definite tensor of thermal conductivity with the particular form dictated by the material symmetry group, and let θ denote temperature. Nanson's formula and energy invariance among configurations lead to relationships among $\bar{\boldsymbol{q}}$, spatial flux $\boldsymbol{q}$, and reference flux $\boldsymbol{Q}$:

(21) $\qquad\qquad \bar{q}^\alpha = \bar{J}\bar{F}^{-1\alpha}_{.a}q^a = \tilde{J}^{-1}\tilde{F}^\alpha_A Q^A = -k^{\alpha\beta}\partial_\beta\theta.$

Heat transfer per unit anholonomic volume is the divergence [6, 7]

$$
\begin{aligned}
(22) \qquad \bar{\nabla}_\alpha \bar{q}^\alpha &\overset{\text{def}}{=} \nabla_\alpha \bar{q}^\alpha + \bar{q}^\alpha \bar{J}(\bar{J}^{-1}\bar{F}^a_{\cdot\alpha})_{:a} = \nabla_\alpha \bar{q}^\alpha + \bar{q}^\alpha \tilde{J}^{-1}(\tilde{J}\tilde{F}^{-1A}_{\cdot\alpha})_{:A} \\
&= \tilde{J}^{-1}\nabla_A Q^A = \bar{J}\nabla_a q^a.
\end{aligned}
$$

Four choices of basis $\{\tilde{\boldsymbol{g}}_\alpha\}$ are considered. In the first case, the anholonomic object is assumed to vanish such that Euclidean position vector $\tilde{\boldsymbol{x}}(X)$ exists:

$$
(23) \qquad \tilde{\boldsymbol{g}}_\alpha = \partial_\alpha \tilde{\boldsymbol{x}}, \quad \Gamma^{\cdot\cdot\alpha}_{\beta\chi} = \tfrac{1}{2}\tilde{g}^{\alpha\delta}\partial_{\{\beta}\tilde{g}_{\delta\chi\}}, \quad (\tilde{J}\tilde{F}^{-1A}_{\cdot\alpha})_{:A} = 0, \quad (\bar{J}^{-1}\bar{F}^a_{\cdot\alpha})_{:a} = 0;
$$

$$
(24) \qquad \bar{\nabla}_\alpha \bar{q}^\alpha = \partial_\alpha \bar{q}^\alpha + \bar{q}^\alpha \partial_\alpha \ln\sqrt{\tilde{g}} = -k^{\alpha\beta}(\partial_\alpha\partial_\beta\theta - \Gamma^{\cdot\cdot\chi}_{\alpha\beta}\partial_\chi\theta).
$$

In the second case, Cartesian intermediate bases $\{\boldsymbol{e}_\alpha\}$ are assigned, but tangent maps need not be integrable:

$$
(25) \qquad \tilde{\boldsymbol{g}}_\alpha \overset{\text{def}}{=} \boldsymbol{e}_\alpha, \quad \tilde{g}_{\alpha\beta} = \delta_{\alpha\beta}, \quad \Gamma^{\cdot\cdot\alpha}_{\beta\chi} = 0, \quad \nabla_\alpha(\cdot) = \partial_\alpha(\cdot);
$$

$$
(26) \qquad \bar{\nabla}_\alpha \bar{q}^\alpha = \partial_\alpha \bar{q}^\alpha + \bar{q}^\alpha \bar{J}\partial_a(\bar{J}^{-1}\bar{F}^a_{\cdot\alpha}) = -k^{\alpha\beta}[\partial_\alpha\partial_\beta\theta + \bar{J}\partial_a(\bar{J}^{-1}\bar{F}^a_{\cdot\alpha})\partial_\beta\theta].
$$

In the third case, $\{\tilde{\boldsymbol{g}}_\alpha\}$ are chosen coincident with reference basis vectors $\{\boldsymbol{G}_A\}$; object $\kappa^{\cdot\cdot\alpha}_{\beta\chi}$, torsion $T^{\cdot\cdot\alpha}_{\beta\chi}$, and curvature from $\Gamma^{\cdot\cdot\alpha}_{\beta\chi}$ all may be nonzero [1]; and

$$
(27) \qquad \tilde{\boldsymbol{g}}_\alpha \overset{\text{def}}{=} \delta^A_\alpha \boldsymbol{G}_A, \quad \Gamma^{\cdot\cdot\alpha}_{\beta\chi} = \tilde{F}^{-1B}_{\cdot\beta}\delta^\alpha_A\delta^C_\chi\Gamma^{\cdot\cdot A}_{BC}, \quad \nabla_\alpha V^\beta = \tilde{F}^{-1A}_{\cdot\alpha}\nabla_A V^B\delta^\beta_B;
$$

$$
(28) \qquad \bar{\nabla}_\alpha \bar{q}^\alpha = -k^{\alpha\beta}[\partial_\alpha\partial_\beta\theta - \tilde{F}^{-1A}_{\cdot\alpha}\delta^B_\beta\delta^\chi_C\Gamma^{\cdot\cdot C}_{AB}\partial_\chi\theta + \tilde{J}^{-1}(\tilde{J}\tilde{F}^{-1A}_{\cdot\alpha})_{:A}\partial_\beta\theta].
$$

In the fourth case, $\{\tilde{\boldsymbol{g}}_\alpha\}$ are chosen coincident with spatial basis vectors $\{\boldsymbol{g}_a\}$; object $\kappa^{\cdot\cdot\alpha}_{\beta\chi}$, torsion $T^{\cdot\cdot\alpha}_{\beta\chi}$, and curvature from $\Gamma^{\cdot\cdot\alpha}_{\beta\chi}$ all may be nonzero [1]; and

$$
(29) \qquad \tilde{\boldsymbol{g}}_\alpha \overset{\text{def}}{=} \delta^a_\alpha \boldsymbol{g}_a, \quad \Gamma^{\cdot\cdot\alpha}_{\beta\chi} = \bar{F}^b_{\cdot\beta}\delta^\alpha_a\delta^c_\chi\Gamma^{\cdot\cdot a}_{bc}, \quad \nabla_\alpha V^\beta = \bar{F}^a_{\cdot\alpha}\nabla_a V^b\delta^\beta_b;
$$

$$
(30) \qquad \bar{\nabla}_\alpha \bar{q}^\alpha = -k^{\alpha\beta}[\partial_\alpha\partial_\beta\theta - \bar{F}^a_{\cdot\alpha}\delta^b_\beta\delta^\chi_c\Gamma^{\cdot\cdot c}_{ab}\partial_\chi\theta + \bar{J}(\bar{J}^{-1}\bar{F}^a_{\cdot\alpha})_{:a}\partial_\beta\theta].
$$

The second case (Cartesian) is most common and presumably most practical for materials of arbitrary anisotropy; the third or fourth cases may prove useful for structures with curved shapes and hexagonal or isotropic symmetry.

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